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SOLUTION OF THE TRANSPORT EQUATION BY  $S_n$  APPROXIMATIONS

Report written by:  
Bengt G. Carlson

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## PHYSICS &amp; MATHEMATICS

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## ABSTRACT

A method for reducing the Transport Equation, an integro-differential equation, to a set of ordinary differential equations is introduced. The reduction is applied to the case of spherical symmetry, and numerical methods for solving the resulting set of equations are developed. The stationary, as well as the time-dependent, equations are considered in detail.

1. Introduction. The present report describes, in brief outline, a new approach to the solution of a large class of diffusion problems. This approach, referred to here as the  $S_n$  Method, is, in generality of application, comparable to the well known Spherical Harmonic Method.

The equation which represents the mathematical formulation of diffusion problems involves, in general, both an integral operator and a first-order partial differential operator. In neutron diffusion work, it is usually referred to as the Transport Equation. Exact analytical solutions are obtainable only in the very simplest cases. Many approximate methods have, therefore, been proposed and explored. Some of these, including the  $S_n$  Method, can be used to obtain approximate analytical solutions if sufficiently simple problems are considered. In general, regardless of method, one has to resort to numerical methods, usually difference techniques. Here the  $S_n$  Method seems to offer many advantages over earlier methods.

2. The Transport Equation. For the case of spherical geometry, one neutron velocity group, and isotropic scattering the Transport Equation has the following form:

$$(1) \quad \left[ \mu D_r + \frac{1 - \mu^2}{r} D_\mu + \sigma_k \right] N(r, \mu) = \sigma_k c_k N(r).$$

$N(r)$  is given by (2), and the bracket denotes a partial differential operator.  $N(r, \mu)$  represents the neutron flux (neuts/cm<sup>2</sup>sec) at the radial distance  $r$  (cm) in the direction  $\theta$  ( $\mu = \cos \theta$ ) with respect to the  $r$ -direction. The parameters  $\sigma_k$  (colls/neut·cm) and  $c_k$  (neuts/coll) describe the media which make up the sphere, and are regarded as step functions of  $r$ . For the central sphere  $k$  equals one, for the first shell  $k$  equals two, etc. The average flux  $N(r)$  is defined by:

$$(2) \quad N(r) = \frac{1}{2} \int_{-1}^1 N(r, \mu) d\mu.$$

Equation (1) also implies a time-independent situation and the absence of separate neutron sources. Generalizations of this case are discussed in Sections 10 and 11.

3. Definition of  $S_n$  Approximations. We divide the  $\mu$ -interval  $(-1, 1)$  into  $n$  intervals  $(\mu_{j-1}, \mu_j)$ ,  $j = 1, 2, \dots, n$ ,  $\mu_0 = -1$ ,  $\mu_n = 1$ , and approximate  $N(r, \mu)$  by  $n$  connected straight line segments as follows:

$$(3) \quad N(r, \mu) = N(r, \mu_{j-1}) + \frac{\mu - \mu_{j-1}}{\mu_j - \mu_{j-1}} \left[ N(r, \mu_j) - N(r, \mu_{j-1}) \right],$$

where  $\mu_{j-1} \leq \mu \leq \mu_j$ ,  $j = 1, 2, \dots, n$ . The integro-differential equation (1) can now be reduced to a system of  $n + 1$  ordinary differential equations in the functions  $N(r, \mu_j)$ ,  $j = 0, 1, \dots, n$ .

The reduction is accomplished by:

(A) substituting  $\mu = -1$  directly in (1), obtaining one equation, and by:



(B) substituting (3) in (1) and then integrating both sides of (1) over  $\mu$  from  $\mu = \mu_{j-1}$  to  $\mu = \mu_j$ ,  $j = 1, 2, \dots, n$ , thus obtaining  $n$  additional equations.

4. The  $S_n$  Equations. Performing the above operations we obtain the following general  $S_n$  equations:

$$(4) \quad \left[ -D_r + \sigma_k \right] N(r, -1) = \sigma_k c_k N(r), \text{ and}$$

$$(5) \quad \left[ a_2 D_r + \frac{b}{r} + 3\sigma_k \right] N(r, \mu_j) + \left[ a_1 D_r - \frac{b}{r} + 3\sigma_k \right] N(r, \mu_{j-1}) = 6\sigma_k c_k N(r),$$

where  $a_2 = 2\mu_j + \mu_{j-1}$ ,  $a_1 = \mu_j + 2\mu_{j-1}$ , and

$b = 2(3 - \mu_j^2 - \mu_j \mu_{j-1} - \mu_{j-1}^2) / (\mu_j - \mu_{j-1})$ . For reasons which we shall not go into here we prefer to divide the  $\mu$ -interval  $(-1, 1)$  into an even number of intervals of equal length.

5. The  $S_2$  Equations: We have two intervals:  $(-1, 0)$  and  $(0, -1)$ , and denote  $N(r, -1)$ ,  $N(r, 0)$  and  $N(r, 1)$  by  $\overleftarrow{N}(r)$ ,  $\overline{N}(r)$ , and  $\overrightarrow{N}(r)$ , respectively. From (4) and (5) we obtain the following three differential equations:

$$(6) \quad \left\{ \begin{array}{l} (-D_r + \sigma_k) \overleftarrow{N}(r) = \sigma_k c_k N(r), \\ (-D_r + \frac{4}{r} + 3\sigma_k) \overline{N}(r) = (2D_r + \frac{4}{r} - 3\sigma_k) \overleftarrow{N}(r) + 6\sigma_k c_k N(r), \\ (2D_r + \frac{4}{r} + 3\sigma_k) \overrightarrow{N}(r) = (-D_r + \frac{4}{r} - 3\sigma_k) \overline{N}(r) + 6\sigma_k c_k N(r). \end{array} \right.$$

6. The  $S_4$  Equations: We have four intervals:  $(-1, -1/2)$ ,  $(-1/2, 0)$ ,  $(0, 1/2)$ , and  $(1/2, 1)$ , and denote  $N(r, -1)$ ,  $N(r, -1/2)$ ,  $N(r, 0)$ ,  $N(r, 1/2)$ , and  $N(r, 1)$  by  $\overleftarrow{N}(r)$ ,  $\overleftarrow{M}(r)$ ,  $\overline{N}(r)$ ,  $\overrightarrow{M}(r)$ , and  $\overrightarrow{N}(r)$ , respectively. From (4) and (5) we obtain the following five differential equations:

$$(7) \quad \left\{ \begin{array}{l} (-D_r + \sigma_k) \overleftarrow{N}(r) = \sigma_k c_k N(r), \\ (-2D_r + \frac{5}{r} + 3\sigma_k) \overleftarrow{M}(r) = (\frac{5}{2}D_r + \frac{5}{r} - 3\sigma_k) \overleftarrow{N}(r) + 6\sigma_k c_k N(r), \\ (-\frac{1}{2}D_r + \frac{11}{r} + 3\sigma_k) \overline{N}(r) = (D_r + \frac{11}{r} - 3\sigma_k) \overleftarrow{M}(r) + 6\sigma_k c_k N(r), \\ (D_r + \frac{11}{r} + 3\sigma_k) \overrightarrow{M}(r) = (-\frac{1}{2}D_r + \frac{11}{r} - 3\sigma_k) \overline{N}(r) + 6\sigma_k c_k N(r), \\ (\frac{5}{2}D_r + \frac{5}{r} + 3\sigma_k) \overrightarrow{N}(r) = (-2D_r + \frac{5}{r} - 3\sigma_k) \overrightarrow{M}(r) + 6\sigma_k c_k N(r). \end{array} \right.$$



7. The Numerical Procedure. We let  $N(r)$  in equations (4) and (5) be given initially as identically equal to unity or some other suitable trial function, and go through an iterative procedure to improve  $N(r)$ , each time solving (4) and (5) for the  $n + 1$  flux functions  $N(r, \mu_j)$ , obtaining the next iterate or improved version of  $N(r)$  from equation (8) below:

$$(8) \quad N(r) = \frac{1}{n} \sum_{j=0}^n w_j N(r, \mu_j),$$

where  $w_0 = w_n = 1/2$ , and  $w_j = 1$ ,  $j = 1, 2, \dots, n - 1$ .

A convergence test of some kind is applied to  $N(r)$  and its successor after each iteration. If it leads to the termination of the iterative procedure, we say that (4) and (5) have been solved numerically relative to this particular test.

The necessary stability conditions on the integration of equations (4) and (5) are:

(a) That the first  $\frac{n}{2} + 1$  equations be integrated in the negative  $r$ -direction, i.e., from the outer boundary ( $r = a$ ) inward to the center, and (b) that the remaining  $\frac{n}{2}$  equations be integrated in the opposite or positive direction. If these conditions are not satisfied, the significant figures in the calculation will rapidly be obscured by an accumulation of errors.

For the first group of equations ( $\mu_j \leq 0$ ) the initial conditions are applied at  $r = a$  and given by:  $N(a, \mu_j) = 0$ . For the second group ( $\mu_j > 0$ ) the initial conditions are applied at the origin and given by:

$$(9) \quad D_r [N(0, \mu_j)] = -D_r [N(0, -\mu_j)] = \mu_j \sigma_1 [c_1 N(0) - \overleftarrow{N}(0, -1)]$$

At interfaces one simply changes parameters and applies continuity conditions on the flux functions.

Summed up, the  $S_n$ -Method has the following important features: (A) the  $n + 1$  equations (4) and (5) need not be solved simultaneously, (B) further complications of the left hand sides of (4) and (5) do not require any changes in the basic procedure, and (C) the stability conditions as well as (D) the boundary conditions are simple in formulation and simple to apply.

To perform the numerical integrations a radial mesh must be introduced (if possible defining equal intervals in each medium) in the core and in each of the spherical shells. This gives rise to a set of  $r_i$ 's,  $i = 0, 1, \dots, I$ ,  $r_0 = 0$ ,  $r_I = a$ , where we let  $r_{i_k}$  denote the interfaces,  $k = 1, 2, \dots, K$ ,  $r_{i_K} = r_I = a$ . To assure accuracy in the final result, no interval should be longer than  $1/n$  mean free path units.

8. The  $S_2$  Difference Equations: From (6) we derive the following difference equations if we center the functions at  $i + \frac{1}{2}$  and use two-point formulae for derivatives and averages:

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$$\begin{aligned}
 (10) \left\{ \begin{aligned}
 \overleftarrow{N}_i &= \frac{(2 - \sigma_k \Delta_k) \overleftarrow{N}_{i+1} + \sigma_k \Delta_k c_k (N_i + N_{i+1})}{2 + \sigma_k \Delta_k}, \\
 \overleftarrow{N}_i &= \frac{(\frac{2}{3} - \sigma_k \Delta_k - 4s_{i+1}) \overleftarrow{N}_{i+1} + (\frac{4}{3} - \sigma_k \Delta_k + 4s_{i+1}) \overleftarrow{N}_{i+1} - (\frac{4}{3} + \sigma_k \Delta_k - 4s_i) \overleftarrow{N}_i + 2\sigma_k \Delta_k c_k (N_i + N_{i+1})}{\frac{2}{3} + \sigma_k \Delta_k + 4s_i}, \\
 \overrightarrow{N}_{i+1} &= \frac{(\frac{4}{3} - \sigma_k \Delta_k - 4s_i) \overrightarrow{N}_i + (\frac{2}{3} - \sigma_k \Delta_k + 4s_i) \overrightarrow{N}_i - (\frac{2}{3} + \sigma_k \Delta_k - 4s_{i+1}) \overrightarrow{N}_{i+1} + 2\sigma_k \Delta_k c_k (N_i + N_{i+1})}{\frac{4}{3} + \sigma_k \Delta_k + 4s_{i+1}},
 \end{aligned} \right.
 \end{aligned}$$

where  $\Delta_k = r_{i+1} - r_i$ ,  $s_i = \Delta_k/3r_i$ . The following special formula is used to start the outward integration:

$$(11) \quad \overrightarrow{N}_1 = \frac{4\overleftarrow{N}_0 + \frac{4}{3}\overleftarrow{N}_1 - (\frac{8}{3} - \sigma_1 \Delta_1) \overleftarrow{N}_1}{\frac{8}{3} + \sigma_1 \Delta_1}$$

9. The  $S_4$  Difference Equations: From (7) we obtain, using the same difference technique as for  $S_2$ :

$$\begin{aligned}
 (12) \left\{ \begin{aligned}
 \overleftarrow{N}_i &= \frac{(2 - \sigma_k \Delta_k) \overleftarrow{N}_{i+1} + \sigma_k \Delta_k c_k (N_i + N_{i+1})}{2 + \sigma_k \Delta_k}, \\
 \overleftarrow{M}_i &= \frac{(\frac{4}{3} - \sigma_k \Delta_k - 5s_{i+1}) \overleftarrow{M}_{i+1} + (\frac{5}{3} - \sigma_k \Delta_k + 5s_{i+1}) \overleftarrow{N}_{i+1} - (\frac{5}{3} + \sigma_k \Delta_k - 5s_i) \overleftarrow{N}_i + 2\sigma_k \Delta_k c_k (N_i + N_{i+1})}{\frac{4}{3} + \sigma_k \Delta_k + 5s_i}, \\
 \overleftarrow{N}_i &= \frac{(\frac{1}{3} - \sigma_k \Delta_k - 11s_{i+1}) \overleftarrow{N}_{i+1} + (\frac{2}{3} - \sigma_k \Delta_k + 11s_{i+1}) \overleftarrow{M}_{i+1} - (\frac{2}{3} + \sigma_k \Delta_k - 11s_i) \overleftarrow{M}_i + 2\sigma_k \Delta_k c_k (N_i + N_{i+1})}{\frac{1}{3} + \sigma_k \Delta_k + 11s_i}, \\
 \overrightarrow{M}_{i+1} &= \frac{(\frac{2}{3} - \sigma_k \Delta_k - 11s_i) \overrightarrow{M}_i + (\frac{1}{3} - \sigma_k \Delta_k + 11s_i) \overleftarrow{N}_i - (\frac{1}{3} + \sigma_k \Delta_k - 11s_{i+1}) \overleftarrow{N}_{i+1} + 2\sigma_k \Delta_k c_k (N_i + N_{i+1})}{\frac{2}{3} + \sigma_k \Delta_k + 11s_{i+1}}, \\
 \overrightarrow{N}_{i+1} &= \frac{(\frac{5}{3} - \sigma_k \Delta_k - 5s_i) \overrightarrow{N}_i + (\frac{4}{3} - \sigma_k \Delta_k + 5s_i) \overrightarrow{M}_i - (\frac{4}{3} + \sigma_k \Delta_k - 5s_{i+1}) \overrightarrow{M}_{i+1} + 2\sigma_k \Delta_k c_k (N_i + N_{i+1})}{\frac{5}{3} + \sigma_k \Delta_k + 5s_{i+1}}
 \end{aligned} \right.
 \end{aligned}$$

and the following special formulae for the outward integrations:

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$$(13) \quad \left\{ \begin{aligned} \vec{M}_1 &= \frac{2\vec{N}_0 + \frac{20}{3}\vec{N}_1 - \left(\frac{13}{3} - \sigma_1\Delta_1\right)\vec{M}_1}{\frac{13}{3} + \sigma_1\Delta_1}, \\ \vec{N}_1 &= \frac{6\vec{N}_0 - \left(\frac{10}{3} - \sigma_1\Delta_1\right)\vec{N}_1 + \left(\frac{1}{3} + \sigma_1\Delta_1\right)\vec{M}_1 + \left(\frac{1}{3} - \sigma_1\Delta_1\right)\vec{M}_1}{\frac{10}{3} + \sigma_1\Delta_1} \end{aligned} \right.$$

In deriving (10) through (13),  $\Delta$  has been assumed to vary with  $k$  only.

### 10. Generalizations.

(A) Time-dependence: Insert  $\frac{1}{v} D_t$  inside the operator bracket of (4), and  $\frac{3}{v} D_t$  inside the operator brackets of (5), where  $t$  (shakes) is the time variable and  $v$  (cm/shake) is the neutron velocity. The time variable  $t$  is added to all flux functions. Cf Section 11.

(B) Many velocity groups: Attach a subscript  $g$  (for group,  $g = 1, 2, \dots, G$ ) to the flux functions in (4) and (5), and replace  $\sigma_k c_k N(r)$  on the right hand sides by:

$$(14) \quad \sum_{g'=1}^G \sigma_{kg'} c_{kgg'} N_{g'}(r),$$

where  $\sigma_{g'}$  is the inverse mean free path as a function of velocity group, and  $c_{gg'}$  is the number of neutrons transferred to group  $g$  per collision of neutrons of velocity  $v_{g'}$ .

(C) Anisotropic scattering: In the case of linear scattering with the scattering function  $(1/2)(1 + 3\beta_k \mu)$  attached to elastic scattering (other processes assumed to be isotropic),  $\sigma_k c_k N(r)$  in (1) is replaced by:

$$(15) \quad \sigma_k c_k \left[ N(r) + 3\beta_k \mu \mathcal{N}(r) \right],$$

where  $\mathcal{N}(r) = \left(\frac{1}{2}\right) \int_{-1}^1 \mu N(r, \mu) d\mu$ , and  $\beta_k = e_k b_k / c_k$ ;  $e_k$  being the probability of elastic collision. In the Transport Theory the anisotropic case is approximated by an isotropic situation in which  $\sigma_k$  is replaced by  $\sigma_k(1 - c_k \beta_k)$  and  $c_k$  by  $c_k(1 - \beta_k)/(1 - c_k \beta_k)$ .

(D) Source term present: Replace  $\sigma_k c_k N(r)$  in (4) and (5) by  $\sigma_k c_k N(r) + S(r)$ , where  $S(r)$  is the source density (neuts/cm<sup>3</sup> sec). An anisotropic source term can, of course, also be handled by going back to equation (1).

(E) Plane geometry: Drop all terms involving  $1/r$  in the above equations, and let  $r$  be the perpendicular distance from some origin plane.

11. The Time-Dependent Case. We consider again equation (1) but with the time variable  $t$  added. Equations (4) and (5) are then replaced by (17) and (18) below:

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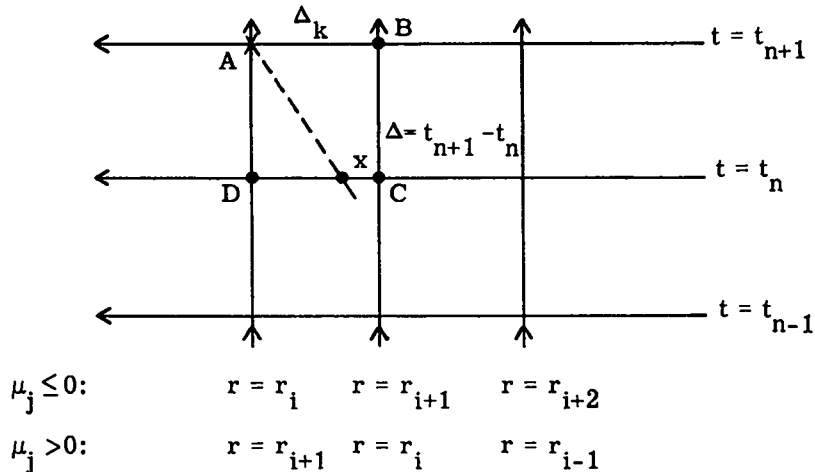


$$(17) \quad \left[ \frac{1}{v} D_t - D_r + \sigma_k \right] N(t, r, -1) = \sigma_k c_k N(t, r)$$

$$(18) \quad \left[ \frac{3}{v} D_t + a_2 D_r + \frac{b}{r} + 3\sigma_k \right] N(t, r, \mu_j) + \left[ \frac{3}{v} D_t + a_1 D_r - \frac{b}{r} + 3\sigma_k \right] N(t, r, \mu_{j-1}) = 6\sigma_k c_k N(t, r).$$

If  $N(t, r_i, \mu_j)$ ,  $i = 0, 1, \dots, I, j = 0, 1, \dots, n$ , and hence  $N(t, r_i)$ , are specified at time  $t = t_0$  one can, by numerical integration of (17) and (18), obtain  $N(t, r_i, \mu_j)$  at later times, say at  $t = t_m$ ,  $m = 1, 2, \dots$ . The following integration method may be used for the purpose. It has these main features: (A) The stability and boundary conditions of Section 7 are left unchanged, and (B) the resulting difference equations are similar to those of Sections 8 and 9.

The integration requires a two-dimensional mesh, here defined by the perpendicular line families  $r = r_i$  and  $t = t_m$ . To integrate (17) and (18) numerically is then equivalent to finding  $N(t, r, \mu_j)$  at A from functions known at B, C, and D. See diagram below:



In the stationary case the difference equations were obtained by averaging derivatives and functions over the line AB. In the time-dependent case we choose the characteristic line or "particle path" Ax as the reference line. The slope of AE equals  $-1/v$  for equation (17) and  $3/va_2$  for equation (18), where  $a_2$  is defined on p. 5. The magnitude of the slope (denoted by  $1/dv$ ) may be greater ( $x$  on DC) or less ( $x$  on BC) than  $\Delta/\Delta_k$ . If a function F has to be evaluated at  $x$ , the following interpolation formula applies:

$$(19) \quad F^x \equiv F(x) = \begin{cases} w_1 F(C) + (1 - w_1) F(D), & w_1 = \frac{dv\Delta}{\Delta k}, \quad dv\Delta \leq \Delta_k \\ w_1 F(C) + (1 - w_1) F(B), & w_1 = \frac{\Delta k}{dv\Delta}, \quad dv\Delta > \Delta_k \end{cases}$$

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12. The  $S_4$  Time-Dependent Difference Equations. Averaging derivatives and functions in (17) and (18) over the appropriate characteristic lines we obtain ( $n = 4$ ):

$$(20) \left\{ \begin{aligned} \overleftarrow{N}_i^{m+1} &= \frac{(2 - \sigma_k \Delta') N_{i+1}^x + \sigma_k \Delta' c_k (N_i^{m+1} + N_{i+1}^x)}{2 + \sigma_k \Delta'}, \\ \overleftarrow{M}_i^{m+1} &= \frac{(\frac{4}{3} - \sigma_k \Delta' - 5s_{i+1}) \overleftarrow{M}_{i+1}^x + (\frac{4}{3} - \sigma_k \Delta' + 5s_{i+1}) \overleftarrow{N}_{i+1}^x - (\frac{4}{3} + \sigma_k \Delta' - 5s_i) \overleftarrow{N}_i^{m+1} + 2\sigma_k \Delta' c_k (N_i^{m+1} + N_{i+1}^x) + \overleftarrow{N}_i^x}{\frac{4}{3} + \sigma_k \Delta' + 5s_i}, \\ \overleftarrow{N}_i^{m+1} &= \frac{(\frac{1}{3} - \sigma_k \Delta' - 11s_{i+1}) \overleftarrow{N}_{i+1}^x + (\frac{1}{3} - \sigma_k \Delta' + 11s_{i+1}) \overleftarrow{M}_{i+1}^x - (\frac{1}{3} + \sigma_k \Delta' - 11s_i) \overleftarrow{M}_i^{m+1} + 2\sigma_k \Delta' c_k (N_i^{m+1} + N_{i+1}^x) + \overleftarrow{M}_i^x}{\frac{1}{3} + \sigma_k \Delta' + 11s_i}, \\ \overleftarrow{M}_{i+1}^{m+1} &= \frac{(\frac{2}{3} - \sigma_k \Delta' - 11t_i) \overleftarrow{M}_i^x + (\frac{2}{3} - \sigma_k \Delta' + 11t_i) \overleftarrow{N}_i^x - (\frac{2}{3} + \sigma_k \Delta' - 11t_{i+1}) \overleftarrow{N}_{i+1}^{m+1} + 2\sigma_k \Delta' c_k (N_i^x + N_{i+1}^{m+1}) + \overleftarrow{N}_i^x}{\frac{2}{3} + \sigma_k \Delta' + 11t_{i+1}}, \\ \overleftarrow{N}_{i+1}^{m+1} &= \frac{(\frac{5}{3} - \sigma_k \Delta' - 5t_i) \overleftarrow{N}_i^x + (\frac{5}{3} - \sigma_k \Delta' + 5t_i) \overleftarrow{M}_i^x - (\frac{5}{3} + \sigma_k \Delta' - 5t_{i+1}) \overleftarrow{M}_{i+1}^{m+1} + 2\sigma_k \Delta' c_k (N_i^x + N_{i+1}^{m+1}) + \overleftarrow{M}_i^x}{\frac{5}{3} + \sigma_k \Delta' + 5t_{i+1}}, \end{aligned} \right.$$

where  $s_i = \frac{\Delta'}{3r_i}$ ,  $s_{i+1} = \frac{\Delta'}{3(r_i + \Delta')}$ ,  $t_{i+1} = \frac{\Delta'}{3r_{i+1}}$ , and  $t_i = \frac{\Delta'}{3(r_{i+1} - \Delta')}$ . If  $dv\Delta \leq \Delta_k$ , then  $\Delta' = dv\Delta$ ,  $w_1 = dv\Delta/\Delta_k$ ,  $F'_i = \frac{1}{3} w_1 (F_{i+1}^m - F_i^m)$ ; and if  $dv\Delta > \Delta_k$ , then  $\Delta' = \Delta_k$ ,  $w_1 = \Delta_k/dv\Delta$ ,  $F'_i = \frac{1}{3} [w_1 (F_{i+1}^m - F_i^m) + (1 - w_1) (F_{i+1}^{m+1} - F_i^{m+1})]$ , where  $d = 1$  for  $\overleftarrow{N}$ ,  $\frac{2}{3}$  for  $\overleftarrow{M}$ ,  $\frac{1}{6}$  for  $\overleftarrow{N}$ ,  $\frac{1}{3}$  for  $\overleftarrow{M}$ , and  $\frac{5}{6}$  for  $\overleftarrow{N}$ .

Since  $N_i^{m+1}$  is not known, we let  $N_i^{m+1} = N_i^m$  for all  $i$ , solve the above equations to obtain a first approximation to  $N_i^{m+1}$ , and repeat the procedure with these values of  $N_i^{m+1}$  to obtain a final set. If the time intervals are large, the advance to  $t = t_{m+1}$  may require more than one iteration.

In the case under consideration,  $N(r, \mu)$  is linear in  $\mu$  for small values of  $r$ . We may, therefore, let  $\overleftarrow{M}_i^{m+1} = 2\overleftarrow{N}_1^{m+1} - \overleftarrow{M}_1^{m+1}$ , and  $\overleftarrow{N}_i^{m+1} = 2\overleftarrow{N}_1^{m+1} - \overleftarrow{N}_1^{m+1}$ . The formulae corresponding to (13), although possibly more accurate, would be very complicated.

The methods described in this report, extended to the many-group case, have been coded for the IBM Type 701 electronic calculator and successfully applied to a variety of neutron diffusion problems. The tables at the end of this report compare  $S_2$  and  $S_4$  with other methods in a series of simple critical mass problems.

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TABLE 1

## UNTAMPED SPHERES

Critical radius (a) in mean free path units

c	EEM <sup>1</sup>	SWM <sup>2</sup>		S <sub>4</sub> Approx.		S <sub>2</sub>
	a	a	% error <sup>3</sup>	a	% error <sup>3</sup>	a
1.1	4.873	4.722	-3.1	4.852	-.4	4.796
1.2	3.172	3.057	-3.6	3.156	-.5	3.095
1.3	2.425	2.331	-3.9	2.411	-.6	2.354
1.4	1.985	1.906	-4.0	1.971	-.7	1.920
1.5	1.690	1.621	-4.1	1.675	-.9	1.630
1.6	1.476	1.415	-4.1	1.463	-.9	1.420
1.7	1.312	1.258	-4.1	1.300	-.9	1.260
1.8	1.183	1.134	-4.1	1.171	-1.0	1.134
1.9	1.078	1.033	-4.2	1.066	-1.1	1.032
2.0	.990	.949	-4.1	.980	-1.0	.947
2.5	.707	.678	-4.1	.699	-1.1	.674
3.0	.551	.529	-4.0	.545	-1.1	.524

1. Extrapolated Endpoint Method, known to be in error by about -0.1% (LA-258).
2. Serber-Wilson Method (LA-234, 247, 756).
3. Compared to EEM.

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TABLE 2

## TAMPED SPHERES

Equal mean free path in core and tamper

Critical core radius (a) in m.f.p. units

c core	c tamper	Tamper thick- ness m.f.p.	EEM	SWM		MDM <sup>1</sup>		S <sub>4</sub> Approx.		S <sub>2</sub>
			a	a	% error <sup>2</sup>	a	% error <sup>2</sup>	a	% error <sup>2</sup>	a
1.3	1.00	1.5	1.896	1.855	-2.2	1.868	-1.5	1.878	-.9	1.816
1.3	.95	1.5	1.973	1.931	-2.1	1.938	-1.8	1.959	-.7	1.894
1.3	.80	1.5	2.132	2.086	-2.2	2.076	-2.6	2.119	-.6	2.056
1.7	1.00	1.5	1.044	1.027	-1.6	1.001	-4.1	1.031	-1.2	.988
1.7	.95	1.5	1.075	1.058	-1.6	1.028	-4.4	1.063	-1.1	1.018
1.7	.80	1.5	1.148	1.127	-1.8	1.085	-5.5	1.136	-1.0	1.092
1.3	1.00	3.0	1.752	1.718	-1.9	1.722	-1.7	1.736	-.9	1.675
1.3	.95	3.0	1.886	1.849	-2.0	1.849	-2.0	1.869	-.9	1.808
1.3	.80	3.0	2.108	2.062	-2.2	2.049	-2.8	2.093	-.7	2.033
1.7	1.00	3.0	.992	.977	-1.5	.947	-4.5	.982	-1.0	.940
1.7	.95	3.0	1.043	1.026	-1.6	.993	-4.8	1.032	-1.1	.989
1.7	.80	3.0	1.137	1.117	-1.8	1.073	-5.6	1.125	-1.1	1.083

1. Modified Diffusion Method, similar to EEM but with interface correction only at the outside boundary.
2. Compared to EEM, known to be in error by about -0.1%.

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TABLE 3

## TAMPED SPHERES

Core m.f.p. equal to  $3/2$  of tamper m.f.p.

Critical core radius (a) in m.f.p. units

c core	c tamper	Tamper thick- ness m. f. p.	"Exact" <sup>1</sup> a	SWM		MDM		S <sub>4</sub> Approx.		S <sub>2</sub> a
				a	% error <sup>2</sup>	a	% error <sup>2</sup>	a	% error <sup>2</sup>	
1.3	1.00	1.5	1.999	1.960	-2.0	2.027	1.4	1.981	-.9	1.910
1.3	.95	1.5	2.057	2.017	-1.9	2.083	1.3	2.040	-.8	1.971
1.3	.80	1.5	2.184	2.135	-2.2	2.193	.4	2.167	-.8	2.099
1.7	1.00	1.5	1.103	1.088	-1.4	1.116	1.2	1.091	-1.1	1.044
1.7	.95	1.5	1.125	1.110	-1.3	1.137	1.1	1.114	-1.0	1.068
1.7	.80	1.5	1.180	1.160	-1.7	1.183	.3	1.169	-.9	1.124
1.3	1.00	3.0	1.893	1.860	-1.7	1.921	1.5	1.876	-.9	1.808
1.3	.95	3.0	1.996	1.957	-2.0	2.020	1.2	1.979	-.9	1.909
1.3	.80	3.0	2.165	2.117	-2.2	2.176	.5	2.149	-.7	2.084
1.7	1.00	3.0	1.068	1.053	-1.4	1.076	.7	1.057	-1.0	1.012
1.7	.95	3.0	1.104	1.088	-1.4	1.113	.8	1.093	-1.0	1.048
1.7	.80	3.0	1.174	1.153	-1.8	1.176	.2	1.163	-.9	1.119

1. Obtained by extrapolation from S<sub>2</sub> and S<sub>4</sub> using Table 2 as a guide.
2. Compared to "Exact."

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TABLE 4

## TAMPED SPHERES

Core m.f.p. equal to 3/4 of tamper m.f.p.

Critical core radius (a) in m.f.p. units

c core	c tamper	Tamper thick- ness m. f. p.	"Exact" a	SWM		MDM		S <sub>4</sub> Approx.		S <sub>2</sub>
				a	% error <sup>1</sup>	a	% error <sup>1</sup>	a	% error <sup>1</sup>	a
1.3	1.00	1.5	1.825	1.780	-2.5	1.774	-2.8	1.811	-.8	1.754
1.3	.95	1.5	1.917	1.871	-2.4	1.851	-3.4	1.902	-.8	1.840
1.3	.80	1.5	2.103	2.052	-2.4	2.004	-4.7	2.088	-.7	2.026
1.7	1.00	1.5	.997	.981	-1.6	.928	-6.9	.987	-1.0	.946
1.7	.95	1.5	1.035	1.018	-1.6	.957	-7.5	1.025	-1.0	.983
1.7	.80	1.5	1.122	1.102	-1.8	1.021	-9.0	1.112	-.9	1.070
1.3	1.00	3.0	1.651	1.615	-2.2	1.593	-3.5	1.638	-.8	1.586
1.3	.95	3.0	1.811	1.772	-2.2	1.736	-4.1	1.796	-.8	1.737
1.3	.80	3.0	2.074	2.024	-2.4	1.968	-5.1	2.059	-.7	1.998
1.7	1.00	3.0	.934	.918	-1.7	.862	-7.7	.924	-1.1	.886
1.7	.95	3.0	.995	.979	-1.6	.914	-8.1	.985	-1.0	.945
1.7	.80	3.0	1.109	1.089	-1.8	1.005	-9.4	1.099	-.9	1.058

1. Compared to "Exact."

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